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BUOYANCY WAVES AS A THERMAL PUMP

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The joint effect of thermal conductivity, viscosity, and of an ascending flow on the thermodynamic processes proceeding in buoyancy waves in a compressible medium is analyzed. It is shown that in this case the waves act as a thermal pump that transfers heat from the upper cold layers of air to the lower warmer ones, thus creating and sustaining a negative temperature gradient. This allows one to adequately explain the temperature stratification observed in the atmosphere and in Ranque–Hilsch tubes.

Keywords: *buoyancy waves, thermal pump, temperature stratification, Carnot cycle, Ranque tube.*

Introduction. In a density-stratified fluid in an equilibrium position the surfaces of equal density coincide with the equipotential lines of a mass force, whereas on being perturbed they vibrate about this position. Such vibrations are called buoyancy waves. In application to the atmosphere, when the mass force is represented by the gravity force, these waves are also called internal gravity waves. They have evoked a number of fundamental research contributions [1–6]. In vortex chambers and in Ranque–Hilsch tubes the density stratification of a gas occurs under the action of a centrifugal force and all conditions are created for the appearance of buoyancy waves. Regular pulsations of pressure and velocity in such facilities are observed at all times, but they are interpreted otherwise, that is, as vortex precession, a secondary vortex, or a regular turbulence [7–10].

It is well known that a thermal pump ensures thermal energy transfer from a cold source to a hot one, in so doing consuming a certain quantity of mechanical energy. An ideal thermal pump operates on the reverse Carnot cycle consisting of two adiabats and two isotherms, and it is characterized by a high conversion factor. Usually, such a process is realized with the aid of machines such as a compressor and an expander combined with two heat exchangers on the cold and hot sides. The technical complexity of such facilities makes one seek other techniques for realizing the Carnot cycle. Buoyancy waves provide such a possibility without machines and heat exchangers.

Mathematical Statement of the Problem. We will consider plane vibrations of a density-stratified viscous isothermal heat conducting gas in the field of a constant mass force.

The vibrations of a stratified atmosphere were considered in a great number of research works [1–4]. Below the equations are given which will be needed to describe such vibrations: the first law of thermodynamics (energy equation):

$$du = dq - dW , \tag{1}$$

the equation of state in the general case, where the relationship between the density, pressure, and temperature is expressed as

$$p = \rho RT . \tag{2}$$

In what follows we will use thermodynamic relations for isochoric and isobaric processes:

$$dq = c_v dT , \quad dv = 0 ; \tag{3}$$

$$dq = c_p dT , \quad dp = 0 , \tag{4}$$

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from which it follows that $du = c_v dT$. The above equations allow one to obtain various dependences that will be used in what follows:

$$dq = c_v dT + p d\nu \,, \tag{5}$$

$$dq = c_p dT - \nu dp \,, \tag{6}$$

$$\frac{dq}{T} = c_p \frac{dT}{T} - R \frac{dp}{p} \,, \tag{7}$$

$$\frac{dq}{T} = - c_p \frac{d\rho}{\rho} + c_v \frac{dp}{p} \,. \tag{8}$$

The continuity equation for a compressible medium is written as

$$\frac{\partial \rho}{\partial t} + \rho \nabla \mathbf{v} = 0 \,. \tag{9}$$

The momentum equation for a gas with viscosity μ has the form

$$\rho \frac{d\mathbf{v}}{dt} = - \nabla p + \rho \mathbf{g} + \mu \Delta \mathbf{v} \,. \tag{10}$$

In our case the mass force $\mathbf{g}$ is a constant vector that has one nonzero z component. Eliminating the temperature from Eqs. (5) and (6), we obtain the energy equation:

$$\frac{dp}{dt} = - \frac{c_p p}{c_v \nu} \frac{d\nu}{dt} + \frac{R}{\nu c_v} \frac{dq}{dt} \,,$$

in which the derivative $d\nu/dt$ can be expressed with the aid of Eq. (9). This gives

$$\frac{dp}{dt} = - \frac{c_p}{c_v} p \nabla \mathbf{v} + \frac{R\rho}{c_v} \frac{dq}{dt} \,. \tag{11}$$

Equation (11) should be supplemented with the heat conduction equation that determines the change in the quantity of heat in a volume due to the thermal conductivity of the medium:

$$\rho_0 \frac{dq}{dt} = \lambda \Delta T \,.$$

The system of equations (2), (9)–(11), subject to the latter formula, serves for determining four unknown functions $\mathbf{v}$, p, ρ, and T. The constructive solution of this system is possible only for the case of small vibrations about a certain stationary state. As the latter, we will take an isothermal equilibrium defined by the functions

$$\mathbf{v}_0 = 0 \,, \quad T_0 = \text{const} \,, \quad p_0 = p_\mathrm{s} \exp\left(- \frac{gz}{RT_0} \right), \quad \rho_0 = \rho_\mathrm{s} \exp\left(- \frac{gz}{RT_0} \right).$$

We will represent a perturbed state in the form of sums:

$$\mathbf{v} = \mathbf{v}_0 + a\mathbf{v}_1 \,, \quad p = p_0 + ap_1 \,, \quad T = T_0 + aT_1 \,, \quad q = q_0 + aq_1 \,.$$

For small perturbations, accurate to the values of the second order of smallness, we can obtain a linear system:

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = -\nabla p_1 + \rho_1 \mathbf{g} + \mu \Delta \mathbf{v}_1 , \tag{12}$$

$$p_1 = R\,(\rho_0 T_1 + \rho_1 T_0) , \tag{13}$$

$$\frac{\partial p_1}{\partial t} + w_1 \frac{dp_0}{dz} + \frac{c_p}{c_v} p_0 \nabla \mathbf{v}_1 = \frac{\rho_0 R}{c_v} \frac{\partial q_1}{\partial t} . \tag{14}$$

Instead of the differential equation of energy (14) it is convenient to use the energy integral obtainable from relations (12) and (14).

We will rewrite the momentum equation (10), using the specific volume:

$$\frac{\partial \mathbf{v}_1}{\partial t} = -v \nabla p + \mathbf{g} + \nu \mu \Delta \mathbf{v}_1 . \tag{15}$$

Going over to the perturbed functions, instead of Eq. (12) we have

$$\frac{\partial \mathbf{v}_1}{\partial t} = -v_1 \nabla p_0 - v_0 \nabla p_1 + \mu \nu_0 \Delta \mathbf{v}_1 . \tag{16}$$

On scalar multiplication of this equation by $\rho_0 \mathbf{v}_1$ and of Eq. (14) by $p_1/\rho_0 c^2$ and summation of the results, we obtain

$$\frac{\partial}{2\partial t}\left(\rho_0 \mathbf{v}_1^2 + \frac{p_1^2}{\rho_0 c^2}\right) + \frac{dp_0}{dz} w_1 \left(\rho_0 v_1 + \frac{p_1}{\rho_0 c^2}\right) + \nabla\,(p_1 \mathbf{v}_1) = \frac{R p_1 dq_1}{c^2 c_v dt} + \mu \mathbf{v}_1 \Delta \mathbf{v}_1 . \tag{17}$$

Here $c = \sqrt{\gamma R T_0}$ is the speed of sound in an isothermal atmosphere. The differential relation (8) after integration in the considered approximation can be written in the form

$$\frac{q_1}{T_0} = -\frac{c_p \rho_1}{\rho_0} + \frac{c_v p_1}{p_0} ,$$

which allows us to transform the expression entering into the energy equation:

$$v_1 \rho_0 + \frac{p_1}{\rho_0 c^2} = -\frac{\rho_1}{\rho_0} + \frac{p_1}{\gamma \rho_0 R T_0} = \frac{q_1}{c_p T_0} , \quad \gamma = \frac{c_p}{c_v} .$$

Thus, relation (17) will be written as

$$\frac{\partial}{2\partial t}\left(\rho_0 \mathbf{v}_1^2 + \frac{p_1^2}{\rho_0 c^2}\right) + \frac{dp_0}{dz} \frac{q_1}{c_p T_0} w_1 + \nabla\,(p_1 \mathbf{v}_1) = \frac{R p_1 dq_1}{c^2 c_v dt} + \mu \mathbf{v}_1 \Delta \mathbf{v}_1 . \tag{18}$$

Let us express the vibrations of the vertical velocity component in terms of entropy fluctuations: $s_1 = q_1/T_0$. For this purpose we will use the evident relationship

$$\frac{\partial s_1}{\partial t} + w_1 \frac{ds_0}{dz} = \frac{dq_1}{T_0 dt} .$$

Elimination of w_1 from Eq. (18) yields

$$\frac{1}{2}\frac{\partial}{\partial t}\left[\rho_0\mathbf{v}_1^2+\frac{p_1^2}{\rho_0 c^2}-\frac{s_1^2}{c_p}\frac{dp_0}{dz}\Big/\frac{ds_0}{dz}\right]+\nabla\,(p_1\mathbf{v}_1)=\left[\frac{Rp_1}{c^2 c_\nu}-\frac{s_1}{c_pT_0}\frac{dp_0}{dz}\Big/\frac{ds_0}{dz}\right]\frac{dq_1}{dt}+\mu\mathbf{v}_1\nabla\mathbf{v}_1\,. \tag{19}$$

From the equation of state that can be written as $\nu = RT/p$ and thermodynamic equation (6) we can obtain

$$s_1=\frac{q_1}{T_0}=\frac{c_pT_1}{T_0}-\frac{Rp_1}{p_0}\,. \tag{20}$$

For an isothermal atmosphere $ds_0 = \frac{-Rdp_1}{dp_0}$, whence $\frac{dp_0}{dz}\Big/\frac{ds_0}{dz} = -\frac{p_0}{R}$. This allows the right-hand side of relation (19) to be rearranged as

$$\frac{Rp_1}{c_\nu c^2}-\frac{s_1}{c_pT_0}\frac{dp_0}{dz}\Big/\frac{ds_0}{dz}=\rho_0\frac{T_1}{T_0}\,.$$

If now we express the thermal energy supply intensity with the aid of the heat-conduction equation, then the energy change law will take the following form:

$$\frac{\partial E}{\partial t}+\nabla\,(p_1\mathbf{v}_1)=\lambda\,\frac{T_1\Delta T_1}{T_0}+\mu\mathbf{v}_1\Delta\mathbf{v}_1\,,$$

where

$$E=\frac{1}{2}\left[\rho_0\mathbf{v}_1^2+p_1^2/\rho_0c^2-\left(\frac{dp_0}{dz}\Big/\frac{ds_0}{dz}\right)\frac{s_1^2}{c_p}\right]$$

is the total energy composed of the kinetic, acoustic, and, according to Eckart's terminology [2], of the thermobaric energy. In order to reveal the physical meaning of the third term, we will carry out the inverse transformation in application to an adiabatic process for which $\frac{\partial s_1}{\partial t}+\frac{w_1ds_0}{dz} = 0$, $\frac{\partial\zeta}{\partial t} = -\left(\frac{\partial s_1}{\partial t}\right)\Big/\left(\frac{\partial s_0}{\partial t}\right)$. By definition the buoyancy frequency (the Väisälä–Brunt frequency) [2] is equal to $N = \sqrt{\frac{gds_0}{c_pdz}}$. Using this definition and the well-known relation of hydrostatistics, we find that the thermobaric energy is

$$-\frac{1}{2c_p}\left(\frac{dp_0}{dz}\Big/\frac{ds_0}{dz}\right)s_1^2=\frac{\rho_0}{2}N^2\zeta^2\,.$$

It is seen that for adiabatic vibrations it is quantitatively equal to the potential energy. Using the Prandtl number Pr, which is close to unity for air, the energy equation may be written as

$$\frac{\partial E}{\partial t}+\nabla\,(p_1\mathbf{v}_1)=\lambda\left[\frac{T_1\Delta T_1}{T_0}+\frac{\mathrm{Pr}}{c_p}\mathbf{v}_1\Delta\mathbf{v}_1\right]. \tag{21}$$

Solution of the Problem. The solution of the system of equations (12)–(14), (21) that describes nonadiabatic vibrations of a viscous heat-conducting gas will be sought in the form of sums:

$$T_1 = T_a + \sqrt{\lambda}\left[\frac{T_2(z)}{\sqrt{\lambda}}\right], \quad w_1 = w_a + \sqrt{\lambda}\left[\frac{w_2(z)}{\sqrt{\lambda}}\right], \quad p_1 = p_a + \sqrt{\lambda}\left[\frac{p_2(z)}{\sqrt{\lambda}}\right], \tag{22}$$

where the first terms relate to stationary adiabatic vibrations of a nonviscous, nonconducting gas. In the solution representation selected we have isolated the small factor $\sqrt{\lambda}$. As will be shown below, the expressions in square brackets remain limited for $\lambda \to 0$. Since the adiabatic solution is well known, we will write it in its final form [1]:

$$p_a = \frac{N^2 Z(z)}{m^2 + \Gamma^2 + (N/c)^2}\sqrt{\frac{\rho_0}{\rho_s}}\exp i(kx - \omega t),$$

$$w_a = -i\omega\sqrt{\frac{\rho_s}{\rho_0}}\,\frac{\sin mz}{m}\exp i(kx - \omega t). \tag{23}$$

Here $Z(z) = \cos mz - (\Gamma/m)\sin mz$; $\rho_s = \rho_0(0)$; $N = \sqrt{\gamma - 1}\,g/c$ is the Väisälä–Brunt buoyancy frequency, $\Gamma = (2 - \gamma)g/c^2$ is the Eckart coefficient, and $\gamma = c_p/c_V$; m, k, ω are the wave numbers and the natural frequency that are coupled by the dispersion relation

$$\frac{\omega^4}{c^2} - \omega^2\left[k^2 + m^2 + \frac{N^2}{c^2} + \Gamma^2\right] + k^2N^2 = 0. \tag{24}$$

The equations of thermodynamics (7) and (8) allow us to express temperature and density in terms of the pressure:

$$T_a = p_a/(\rho_0 c_p), \tag{25}$$

$$p_a = p_a/c^2. \tag{26}$$

With the aid of the continuity equation we can determine the horizontal velocity component U_a of adiabatic vibrations. If this is done in the approximation of local incompressibility, which is valid for low-frequency vibrations, we obtain

$$U_a = \omega\sqrt{\frac{\rho_s}{\rho_0}}\,\frac{\cos mz}{k}\exp i(\omega t - kx). \tag{27}$$

The representation of (22) selected by us for nonadiabatic vibrations determines those additions to adiabatic quantities that on the average for the period coincide with the known functions. The overbar will designate the result of averaging over the variable x or, which is the same, over t. Then we may write $\overline{w}_1 = w_2(z)\overline{p}_1 = p_2(z)$, etc.

In order to obtain equations for unknown functions, we substitute Eq. (22) into the system of equations (12)–(14), (21) and average these equations over time. As a result, we obtain the following system of equations:

$$dp_2/dz = -g\rho_2, \tag{28}$$

$$w_2 d\rho_0/dz + \rho_0 dw_2/dz = 0, \tag{29}$$

$$p_2 = R(\rho_0 T_2 + \rho_2 T_0). \tag{30}$$

In deriving them, we have availed ourselves of the fact that the average values of all the parameters for adiabatic vibrations $\overline{w}_a = \overline{p}_a = \overline{\rho}_a = 0$. Moreover, it is taken into account that the nonadiabatic addition is proportional to the square root of thermal conductivity. Consequently, the viscous term omitted in Eq. (7) has the order of smallness $\lambda^{3/2}$.

Now, we will consider the energy equation (21). By virtue of the periodicity of the solution $\overline{\partial E/\partial t} = 0$. Consequently, Eq. (21) with the above-adopted accuracy is reduced to

$$\frac{d\,(w_1p_1)}{dz} = \lambda\left[\overline{T_a\Delta T_a/T_0} + \mathrm{Pr}\,\overline{\mathbf{v}_a\Delta\mathbf{v}_a}/c_p\right] + O\,(\lambda^2)\,.$$

Using the equality $\overline{w_ap_a} = 0$, from relations (25) and (26), we obtain

$$\frac{d}{dz}\,(w_2p_2) = \lambda\left[p_a\Delta\left(\frac{\overline{p}_a}{\rho_0}\right)\frac{1}{c_p^2\rho_0T_0} + \frac{\mathrm{Pr}}{c_p}\,\overline{\mathbf{v}_a\Delta\mathbf{v}_a}\right]. \tag{31}$$

Provided $m >> g/RT_0$, the following approximate relation will be valid:

$$\Delta\left(\frac{p_a}{\rho_0}\right) = \frac{\Delta p_a}{\rho_0} \approx -\,(m^2 + k^2)\,\frac{p_a}{\rho_0}\,.$$

From the dispersion equation (24) it is seen that the large wave number m corresponds to the small frequency ω. Consequently, the above-adopted condition can always be obeyed if we restrict our consideration to low-frequency vibrations. In this case, Eq. (31) is reduced to the form

$$\frac{d}{dz}\,(w_2p_2) = -\,\lambda\,(m^2 + k^2)\left(\frac{\overline{p}_a^2}{c_p^2\rho_0^2T_0} + \frac{\mathrm{Pr}}{c_p}\,\overline{\mathbf{v}}_a^2\right).$$

Substituting the ratios $\frac{w_2}{\sqrt{\lambda}}$ and $\frac{p_2}{\sqrt{\lambda}}$ into the above equation, we can easily see that they satisfy the equation that is independent of λ, as are also Eqs. (28)–(30), thus confirming our earlier assumption. We will transform the left-hand side of the latter relation, using for this purpose the values of the derivatives and functional couplings (28)–(30):

$$\frac{d}{dz}\,(w_2p_2) = \frac{w_2\rho_0gT_2}{T_0}\,. \tag{32}$$

We note that the integral $w_2\rho_0 = w_2(0)\rho_s = \mathrm{const}$ follows from Eq. (29).

The transformations made allow us to express the temperature addition that appeared because of the viscosity and thermal conductivity of the gas:

$$\frac{T_2}{T_0} = -\,\frac{A\,(m^2 + k^2)}{gw_2\,(0)}\left(\frac{\overline{p}_a^2}{c_p\rho_0T_0} + \mathrm{Pr}\,\overline{\mathbf{v}}_a^2\rho_0\right)\left(\frac{\rho_s}{\rho_0}\right), \quad A = \frac{\lambda}{\rho_sc_p}\,. \tag{33}$$

The values of the quantities defined by Eqs. (23) and (27) for the functions p_a and $\mathbf{v}_a$ entering into Eq. (33) allow us to determine their mean square values:

$$\overline{p}_a^2 = \frac{N^4\rho_0\rho_sZ^2\,(z)}{2\,[m^2 + \Gamma^2 + (N/c)^2]}\,, \tag{34}$$

$$\overline{\mathbf{v}}_a^2 = w_a^2 + U_a^2 = \frac{\omega^2\rho_s}{2m^2\rho_0}\left(\sin^2 mz + \frac{m^2}{k^2}\cos^2 mz\right).$$

The solution obtained contains both acoustic waves and the buoyancy waves proper. The latter correspond to low frequencies. In this case, the dispersion relation can be replaced by $(\omega/k)^2 = N^2\left[m^2 + \Gamma^2 + (N/c)^2\right]$; consequently

$$\overline{\mathbf{v}_{\mathrm{a}}^{2}} = \frac{N^2}{2\rho_0\left[m^2 + \Gamma^2 + \left(\dfrac{N}{c}\right)^2\right]}\left(\cos^2 mz + \frac{k^2}{m^2}\sin^2 mz\right)$$

and Eq. (33) takes the form

$$\frac{T_2}{T_0} = -\frac{A\,(m^2 + k^2)\,S^2}{2gw_2\,(0)}\,G\,, \tag{35}$$

where

$$S^2 = \frac{N^2}{m^2 + \Gamma^2 + (N/c_{\mathrm{s}})^2}\,; \quad G = \frac{S^2\,(1-\gamma)}{c^2}\left(\cos^2 mz + \frac{\Gamma^2}{m^2}\sin^2 mz - \frac{\Gamma}{m}\sin^2 mz\right)\exp\left(\frac{gz}{RT_0}\right)$$

$$+ \Pr\left(\cos^2 mz + \frac{k^2}{m^2}\sin^2 mz\right)\exp\left(\frac{gz}{RT_0}\right).$$

To this we should also add the nonperturbed distribution of pressure and density:

$$p_0 = p_{\mathrm{s}}\exp\left(-\frac{gz}{RT}\right), \quad \rho_0 = \rho_{\mathrm{s}}\exp\left(-\frac{gz}{RT_0}\right).$$

Having excluded the density from Eqs. (28) and (30), we obtain an equation for p_2:

$$\frac{dp_2}{dz} + \frac{p_2 g}{RT_0} = \frac{g\rho_0 T_2}{T_0}\,.$$

The solution of this equation without the right-hand side can be included into the zero approximation, whereas the solution of the inhomogeneous equation can be represented by the quadrature:

$$p_2 = \frac{g\rho_{\mathrm{s}}}{T_0}\exp\left(-\frac{gz}{RT_0}\right)\int_0^z T_2\,(\xi)\,d\xi\,, \tag{36}$$

in which the temperature is expressed by formula (35).

To calculate the entropy perturbation, we will use Eq. (20). Having integrated it within small limits, we obtain a relation for perturbed quantities for which we had adopted the representations (22). The operation of averaging allows us to obtain the entropy increment due to the viscosity and thermal conductivity of the gas:

$$s_2 = c_p\frac{T_2}{T_0} - \frac{g}{T_0^2}\int_0^z T_2\,(\xi)\,d\xi\,. \tag{37}$$

Having found the vertical velocity component $w_2 = w_2(0)\exp\,(gz/RT_0)$ from Eq. (29), we complete the definition of all the quantities of interest to us.

In the classical statement of the problem the system of equations (2), (9)–(11) must be supplemented by boundary-value conditions. Following the practice adopted in the theory of buoyancy waves, we have found periodic solutions in a semispace; now we consider what boundary-value conditions the solution obtained will satisfy. In particular, we see that the vertical velocity component on the lower boundary of the region must be different from zero.

Indeed, if we consider the problems for which $w_2(0) = 0$, then from the energy equation (21) we obtain the condition $dE/dt < 0$, signifying that the vibrations are damping and that they cannot be sought in the form of periodic functions (22). Moreover, only a certain value of the boundary value of $w_2(0)$ ensures the existence of stationary vibrations. Equation (32), subject to (33), considered at $z = 0$ makes it possible to obtain the needed value of $w_2(0)$ as a function of other parameters of the problem:

$$w_2(0) = -\frac{A(m^2+k^2)S^2}{2g}\left[\frac{S^2(1-\gamma)}{c_s^2} + \mathrm{Pr}\right]\frac{T_0}{T_2(0)}. \tag{38}$$

We note that $w_0 = 0$ and $w_a = 0$ at $z = 0$; consequently, the boundary value for $w_2(0)$ is the boundary value for the total velocity on the boundary.

Strictly speaking, the temperature and the tangent component of the gas velocity on the boundary must coincide with the values of these quantities for a solid wall, i.e., the boundary must have vibrations in the horizontal plane and periodically change temperature according to the travelling wave law. Otherwise, dynamic and temperature boundary layers appear near the boundary within which the oscillating values of the functions in the gas correlate with the constant values of these functions on the boundary. The thicknesses of the dynamic and temperature boundary layers have an order of $\sqrt{\lambda}$ and, consequently, have small dimensions. Thus, in the entire flow region beyond the thin boundary layer we may use the solutions obtained above.

Analysis of Solution. Let us continue the solution analysis begun above. Consider Eq. (35) that defines the temperature. It consists of the product of an oscillating function of the form $\cos^2 mz$ and a slowly changing function $\exp(gz/RT_0)$. In our adopted condition of the smallness of wavelengths the value of temperature averaged over the variable z is equal to

$$\frac{T_2}{T_0} = -\frac{a_s(m^2+k^2)S^2}{4}gw_2(0)\left[(1-\gamma)\frac{S^2}{c^2} + \mathrm{Pr}\right]\exp\left(\frac{gz}{RT_0}\right).$$

We introduce the notation

$$P = \frac{a_s(m^2+k^2)S^2}{4}gw_2(0)\left[(1-\gamma)\frac{S^2}{c^2} + \mathrm{Pr}\right]$$

and write the formula for the averaged temperature in the form

$$\frac{T_2}{T_0} = -P\exp\left(\frac{gz}{RT_0}\right).$$

The averaged temperature gradient is equal to

$$\frac{dT_1}{dz} = -\frac{Pg}{RT_0}\exp\left(\frac{gz}{RT_0}\right) \approx -\frac{Pg}{RT_0}.$$

As is known, according to the condition of convective stability of a stratified atmosphere the gradient must not exceed the negative value $-g/c_p$. Provided the condition is violated, in the atmosphere a convective flow is induced that decreases the temperature gradient to an admissible value. If we take into account the fact that the temperature and, consequently, its gradient oscillate over the altitude, it becomes clear why the predicted value of 10°C per 1 km turns out to be smaller than the observed 6°C per 1 km of the altitude.

Let us take Eq. (31) and write it in an integral form:

$$w_2(z)p_2(z) - w_2(0)p_2(0) = -\int_0^z Q\,d\xi, \quad Q = -\lambda(m^2+k^2)\left[\frac{\overline{T}_a^2}{T_0} + \frac{\mathrm{Pr}\,\overline{\mathbf{v}}_a^2}{c_p}\right].$$

This expression determines the work of pressure forces which is supplied to the volume considered. Precisely this work compensates the dissipation of mechanical energy and ensures stationary vibrations of a viscous heat-conductivity fluid. The right-hand side determines the equivalent quantity of thermal energy that is evolved in the volume and should be transferred through the boundary.

The value of temperature averaged over the variable z allows one to determine the averaged value of entropy from Eq. (37), if preliminarily it is subjected to the operation of averaging over the variable z. As a result we obtain the dependence

$$\tilde{s} = -P\left[c_p + R\left(1 - \exp\frac{gz}{RT_0}\right)\right],$$

from which it is seen that the entropy acquires a negative increment for the altitudes smaller than

$$z_0 = \frac{RT_0}{g}\ln\left(1 + \frac{c_p}{R}\right).$$

Since for a diatomic gas $c_p = 3.5R$, then

$$z_0 = 3RT_0/2g\ . \tag{39}$$

Now, when the solution of the problem has been obtained, we may follow the dynamics and thermodynamics of the liquid particles in Lagrangian variables. The motion of particles can be determined from the velocity vector. The vertical component is prescribed by formula (23), and the horizontal one by relation (27).

Let us denote the Lagrangian coordinates of particles by ζ and ξ and the coordinates of the point in space near which the particle performs small vibrations by x_0 and z_0. Then

$$\frac{\partial\zeta}{\partial t} = \frac{\omega}{m}\sqrt{\frac{\rho_s}{\rho_0}}\ \sin mz_0 \sin(kx - \omega t)\ ,$$

$$\frac{\partial\xi}{\partial t} = \frac{\omega}{k}\sqrt{\frac{\rho_s}{\rho_0}}\ \cos mz_0 \cos(kx - \omega t)\ .$$

Integration of these equations yields

$$\zeta = z_0 + \frac{1}{m}\sqrt{\frac{\rho_s}{\rho_0}}\ \sin mz_0 \cos\varphi\ , \quad \xi = x_0 - \frac{1}{k}\sqrt{\frac{\rho_s}{\rho_0}}\ \cos mz_0 \sin\varphi\ , \quad \varphi = kx - \omega t\ .$$

Having excluded the parameters φ, we find the trajectory along which the particles move:

$$\frac{(\zeta - z_0)^2}{a^2} + \frac{(\xi - x_0)^2}{b^2} = 1\ , \tag{40}$$

where

$$a = \frac{1}{m}\sqrt{\frac{\rho_s}{\rho_0}}\ \sin mz_0\ ; \quad b = \frac{1}{k}\sqrt{\frac{\rho_s}{\rho_0}}\ \cos mz_0\ .$$

The dynamics of the process is determined by the momentum equation (12). In the approximation adopted by us, the viscous term is calculated from the adiabatic value of velocity and may be considered a known function:

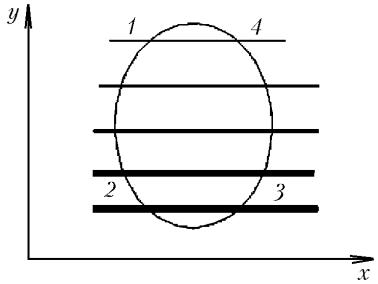

Fig. 1. Trajectories and phases of the motion of a particle: 1) φ = π/4; 2) 3π/4; 3) 5π/4; 4) 7π/4.

$$\rho_0 \frac{d\mathbf{v}_1}{dt} = - \nabla p_1 + \rho_1 \mathbf{g} + \mu \Delta \mathbf{v}_a .$$

Since

$$\Delta \mathbf{v}_a = - (m^2 + k^2)\, \mathbf{v}_a ,$$

then

$$\rho_0 \frac{d\mathbf{v}_1}{dt} = - \nabla p_1 + \rho_1 \mathbf{g} - \mu (m^2 + k^2)\, \mathbf{v}_a .$$

We note that the same equation can be obtained by using the Rayleigh method [2] with the only essential difference that in our case the coefficient at the velocity is the quantity to be calculated which depends on the flow parameters. The thermal processes are determined by the energy equation

$$\frac{\partial p_1}{\partial t} + w_1 \frac{dp_0}{dz} + \gamma p_0 \nabla \mathbf{v}_1 = \frac{\rho_0 R}{c_v} \frac{dq_1}{dt} ,$$

the right-hand side of which satisfies the heat conduction equation $\rho_0 dq_1/dt = -\lambda \Delta T_1$.

Just as in the case of the momentum equation, the perturbation of temperature can be replaced by the adiabatic value. As a result we obtain

$$\rho_0 \frac{dq_1}{dt} = - \lambda (m^2 + k^2)\, T_a . \tag{41}$$

Formula (40) determines the elliptical trajectories along which the isolated particles of air move. Since the wave numbers m and k can be arbitrary, then the values of the half-axes a and b of the ellipsoid depicted in Fig. 1 can also be arbitrary. In this very figure the sequences of phases that correspond to the change of phases of a thermodynamic process are indicated.

Expression (41) determines the intensity of a heat source or of a heat flux depending on the temperature sign. Having expressed the adiabatic temperature in terms of the pressure with the aid of analytical representation, we represent the heat source function in the form

$$\Phi (\varphi) = - \lambda (m^2 + k^2)\, T_a = - \Phi_0 \cos \varphi , \quad \Phi_0 = \lambda R (m^2 + k^2)\, S^2 \sqrt{\frac{\rho_s}{\rho_0}} \frac{Z(z_0)}{c_v c_p} .$$

On the upper portion of the trajectory over which the particle passes from point 4 to point 1, the phase changes in the interval $-\pi/4 \le \varphi \le \pi/4$. In this case the function Φ(φ) has a negative value and, consequently, the

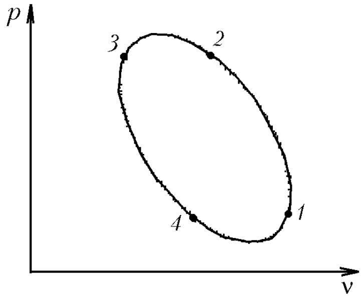

Fig. 2. p–ν diagram of the process.

process proceeds with heat absorption, and on the lower portion from 2 to 3 ($3\pi/4 \le \varphi \le 5\pi/4$) the function $\Phi(\varphi)$ is positive, and the process proceeds with heat release. At the same time, vertical portions 1–2 and 3–4 correspond to a minimum value of the heat source density, since at the points $\varphi = \pi/2$ and $3\pi/2$ the heat flux vanishes. Consequently, over these portions the gas particle is compressed or expanded almost following an adiabat. We note that on the upper portion the particle absorbs the low-potential heat at a temperature $T_0 - |T_a|$, whereas on the lower portion it evolves a high-potential heat at a temperature $T_0 + |T_a|$. Precisely this mechanism is considered to be responsible for the decrease in the entropy and sustainment of a temperature gradient in the atmosphere.

In Fig. 2 the p–ν diagram of the cycle is depicted in which the considered gas particle participates. As is said above, along the vertical portions 1–2 and 3–4 the processes progress almost following the adiabatic law. The horizontal portions 4–1 correspond to expansion with heat supply, whereas portion 2–3 corresponds to compression with heat removal. As is known, such a cycle corresponds to the reverse Carnot cycle and is realized in thermal pumps.

Formula (39) determines the altitude to which the action of the thermal pump propagates. The substitution of the atmosphere parameters into it makes it possible to obtain the value z = 10.8 km. It coincides with the altitude of the troposphere within which the temperature follows the linear law and then over tens of kilometers preserves its value intact [1, 2].

It was a widespread erroneous opinion that the temperature gradient was sustained by the radiation balance, that is, the earth's surface absorbs solar energy due to which a high temperature of the lower atmosphere layers is sustained, whereas the radiation of the upper layers into the outer space ensures their low temperature. In [11], calculation of temperature distribution in the atmosphere is given. It is determined by radiation absorption and reemission of heat. The calculation shows that already beginning from an altitude of 3 km the radiation mechanism is unable to create a temperature gradient and, more so, it cannot explain the temperature gradient jump observed on the upper boundary of the troposphere. But a thermal pump that at each level pumps heat from a colder upper level towards a hotter lower one and ceases its work at an altitude of 10.8 km can explain this jump. Indeed, if the performance of a thermal pump turns out to be equal to the gradient heat flux, the heat transfer from level to level will be equal to zero irrespective of the temperature gradient and of the thermal conductivity of air. At the same time, above the troposphere, where the thermal pump ceases its work, identical temperature is sustained at various altitudes due to the thermal conductivity.

As is known [12], the temperature gradient is determined by the condition of hydrostatic stability of the atmosphere and is equal to 10^{o}C per 1 km. The discrepancy between this value and that observed, equal to 6^{o}C per 1 km, can be explained by the spatial variable-oscillating law of change in the local temperature, as follows from Eq. (35), i.e., the amplitude value of the temperature gradient is 10^{o}C per 1 km, whereas its average value is only 6^{o}C per 1 km.

In the Ranque–Hilsch tubes there are always velocity and pressure fluctuations that are never related to temperature stratification. However, in the experimental work [9] data are given according to which a decrease in the level of pressure fluctuations in the Ranque tube by 30 dB led to a decrease in the temperature difference by 30^{o}C, which may serve as an argument in favor of the adopted model of the process.

Conclusions. According to the present author's opinion, the results obtained not only allow one to explain the observed physical effects such as the temperature stratification in the atmosphere and in the Ranque tubes, but also to create a thermal pump of simple construction with a high conversion factor.

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NOTATION

A, thermal diffusivity of a gas, m^2/sec; a, b, dimensionless amplitudes; c, speed of sound, m/sec; c_p, c_v, heat capacities at constant pressure and volume, J/(kg·^{o}C); g, gravity force, m^2/sec; m, k, wave numbers, 1/m; N, buoyancy frequency, m^2/sec; p, pressure, N/m^2; q, supplied heat, J/kg; R, gas constant, J/(kg·^{o}C); S, velocity of the propagation of waves, m/sec; s, entropy, J/(kg·^{o}C); T, temperature, ^{o}C; t, time, sec; u, internal energy of a gas, J/kg; $\mathbf{v}$, velocity vector, m/sec; W, work of outer forces, W; w, vertical velocity component, m/sec; x, y, z, coordinates of space, m; Γ, Eckart's coefficient, 1/m; $\gamma = c_p/c_v$, ratio of heat capacities; λ, thermal conductivity, W/(m·^{o}C); μ, dynamic viscosity, kg/(m·sec); $\nu = 1/\rho$, specific volume, m^3/kg; ρ, density, kg/m^3; ξ, ζ, coordinates of a particle, m. Subscripts: a, adiabatic process; 0, initial stationary state; 1, perturbed state; 2, averaged values of parameters; s, boundary values of parameters.

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